HIGH-RESOLUTION COMBINED COMPACT SCHEMES FOR HYPERBOLIC CONSERVATION LAWS

Katsuya Ishii*, Kazuya Matsuoka, Kyota Hattori and Shintaro Yamamoto

*Nagoya University,
ITCF, Furo-cho, Chikusa-ku, Nagoya 464-8601, Japan
e-mail: ishii@itc.nagoya-u.ac.jp

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Abstract. We develop the conservative upwind combined compact difference scheme. Using this scheme we propose a conservative hybrid combined compact-WENO scheme to obtain the high accurate solution of hyperbolic conservation laws. The conservative upwind combined compact difference scheme is coupled on to a WENO scheme in the hybrid scheme. This hybrid scheme is applied to solve the Euler equations in one and two dimensions. Numerical results show that the proposed scheme has high resolution properties. We also discuss a nonlinear scheme with the combined compact difference for hyperbolic conservation laws.

1 INTRODUCTION

In simulations of turbulence and aero-acoustics, a high-resolution scheme is required to compute the various scales of the flows accurately. Lele\textsuperscript{3} proposed the Compact Difference (CD) scheme, which yields high resolution and is able to capture weak waves such as turbulent and/or acoustic fluctuations. Chu&Fan\textsuperscript{5} and Nihei&Ishii\textsuperscript{6} were proposed the Combined Compact Difference (CCD) Scheme. CCD schemes use higher derivatives at 3-point stencil and have higher resolution than CD schemes with the same accuracy. However, if the CD scheme and/or the CCD scheme are applied to problems that contain discontinuities, non-physical oscillations (Gibbs phenomena) appear in the numerical solution.

If flow fields involve shock waves, a shock-capturing scheme should be used to resolve the discontinuities without oscillation. Liu et al.\textsuperscript{4} proposed Weighted Essentially Non-Oscillatory (WENO) scheme, and it demonstrates very promising shock-capturing capabilities. However, the resolution of the WENO scheme is lower than the CD and CCD scheme, and it can lead to a significant damping of weak waves.

Pirozzoli\textsuperscript{1} combined the advantages of both the CD scheme and the WENO scheme, and proposed a conservative hybrid compact-WENO (HCD) scheme to compute shock-turbulence interaction problems efficiently. The algorithm is based on the CD scheme of the conservation form to compute the smooth part of the flow field, which is coupled with the WENO scheme to capture the discontinuity.

Deng&Zhang\textsuperscript{2} introduced the CD scheme to the weighted computation, and proposed the Weighted Compact Nonlinear Scheme (WCNS) which does not need the switching and can resolve the discontinuity without oscillatory. WCNS does not have as high resolution as CD
schemes.

In this paper, we propose a conservative hybrid combined compact-WENO (HCCD) scheme to obtain high resolution for the solution of hyperbolic conservation laws. First, we develop the combined compact difference of the conservation form. HCCD improves the accuracy and resolution of the HCD. HCCD is applied the Euler equations in one and two dimensions in order to examine the performance of HCCD. In addition, in chapter 4, we modify a Weighted Compact Nonlinear Scheme (WCNS) of Deng&Zhang\textsuperscript{2} and propose a Weighted Combined Compact Nonlinear Scheme (WCCNS) in the finite-volume representation. A new scheme is applied the two dimensional Euler equations in order to denote the extensibility of the combined compact difference.

2 NUMERICAL FORMULATION

2.1 A conservative Combined Compact Difference scheme

Consider the scalar hyperbolic conservation low given by

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0.$$  \hspace{2cm} (1)

When we use the grid system of N grid points with uniform spacing $h$, $u_j, f_j$ and $f'_j$ denote the values of the functions $u, f(u)$ and its 1st derivative $f'(u)$ at $j$-th grid point $x_j$.

Eq.(1) is spatially discretized in conservation form

$$\frac{\partial u_j}{\partial t} = -\frac{\tilde{f}_{j+1/2} - \tilde{f}_{j-1/2}}{h} \quad (j = 1, 2, \cdots, N)$$  \hspace{2cm} (2)

where $\tilde{f}_{j+1/2}$ is the value of the numerical flux function at the intermediate node $x_{j+1/2}$. If

$$\frac{1}{h}(\tilde{f}_{j+1/2} - \tilde{f}_{j-1/2}) = f'_j + O(h^k),$$  \hspace{2cm} (3)

this scheme is k-th order accuracy in space.

Pirozzoli\textsuperscript{1} proposed a conservative compact scheme whereby the reconstructions of $\tilde{f}_{j+1/2}$. We propose a conservative combined compact scheme based on CCD by the reconstructions of $\tilde{f}_{j+1/2}$.

For $k = 7$ in Eq.(3), the numerical flux function in Eq.(2) can be computed by using the 7th-order conservative upwind combined compact scheme

$$\begin{pmatrix} A & E \\ B & R \end{pmatrix} \begin{pmatrix} \tilde{f}_{j-1/2} \\ \tilde{f}'_{j-1/2} \end{pmatrix} + \begin{pmatrix} \tilde{f}_{j+1/2} \\ \tilde{f}'_{j+1/2} \end{pmatrix} + \begin{pmatrix} \tilde{f}_{j+3/2} \\ \tilde{f}'_{j+3/2} \end{pmatrix} = \begin{pmatrix} \tilde{f}_{j+1/2} \\ \tilde{f}'_{j+1/2} \end{pmatrix}$$  \hspace{2cm} (4)

where
\[
A = \begin{pmatrix}
\frac{5}{8} & \frac{h}{8} \\
\frac{9}{2} & \frac{1}{8} \\
\frac{8h}{8} & \frac{1}{8}
\end{pmatrix}, \quad
B = \begin{pmatrix}
\frac{23}{72} & \frac{h}{24} \\
\frac{9}{8} & \frac{1}{8} \\
\frac{8h}{8} & \frac{1}{8}
\end{pmatrix}, \quad
E = \begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix},
\]

\[
R_{j+1/2} = \begin{pmatrix}
-\frac{1}{54} f_{j-1} + \frac{133}{108} f_j + \frac{79}{108} f_{j+1} \\
\frac{1}{h} (-3 f_j + 3 f_{j+1})
\end{pmatrix}
\]

When $\partial f / \partial x > 0$, the evaluation of the numerical flux at the left boundary at intermediate $x_{1/2}$ requires the knowledge of the solution in an additional ghost node $x_0$. Given by the function value $f_0$ at the ghost node $x_0$, at the left boundary we have

\[
E \begin{pmatrix}
\tilde{f}_{j+1/2} \\
\tilde{f}_{j-1/2}
\end{pmatrix} + C_1 \begin{pmatrix}
\tilde{f}_{j+3/2} \\
\tilde{f}_{j-3/2}
\end{pmatrix} = R_{j+1/2}
\]

(5)

where

\[
C_1 = \begin{pmatrix}
\frac{3}{2} & -\frac{3}{4} \\
-\frac{5}{2h} & \frac{1}{4}
\end{pmatrix}, \quad
R_{j+1/2} = \begin{pmatrix}
\frac{1}{16} f_0 + \frac{139}{48} f_i - \frac{23}{48} f_2 + \frac{1}{48} f_3 \\
\frac{1}{h} \left( -\frac{23}{48} f_0 - \frac{199}{48} f_i - \frac{107}{48} f_2 - \frac{5}{48} f_3 \right)
\end{pmatrix}
\]

and at the right boundary

\[
D_1 \begin{pmatrix}
\tilde{f}_{N-1/2} \\
\tilde{f}_{N+1/2}
\end{pmatrix} + E \begin{pmatrix}
\tilde{f}_{N-1/2} \\
\tilde{f}_{N+1/2}
\end{pmatrix} = R_{N+1/2}
\]

(6)

where

\[
D_1 = \begin{pmatrix}
\frac{2}{3} & 2h \\
\frac{35}{9h} & \frac{37}{3}
\end{pmatrix}, \quad
R_{N+1/2} = \begin{pmatrix}
-\frac{1}{36} f_{N-3} + \frac{11}{36} f_{N-2} - \frac{97}{36} f_{N-1} + \frac{49}{12} f_N \\
\frac{1}{h} \left( -\frac{23}{108} f_{N-3} + \frac{247}{108} f_{N-2} - \frac{2075}{108} f_{N-1} + \frac{53}{4} f_N \right)
\end{pmatrix}
\]

This scheme becomes 6th-order accuracy at the boundaries. When $\partial f / \partial x < 0$, the boundary closure is the symmetrical form to the above boundary closures.

### 2.2 Stability analysis

For investigating the stability we consider a simple linear advection equation
\[
\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \quad f = au, \quad (7)
\]

Using a compact matrix notation, the proposed CCD with the boundary closures is represented by

\[
P \bar{F} = Qf + q
\]

where

\[
\bar{F} = \left[ \bar{f}_{1/2}, \bar{f}_{3/2}, \ldots, \bar{f}_{N+1/2} \right]^T, \quad f = \left[ f_1, f_2, \ldots, f_N \right]^T, \quad q = \left[ q_1, q_2, 0, \ldots, 0 \right]^T,
\]

\[
\bar{f}_{j+1/2} = \left( \tilde{f}_{j+1/2}, \tilde{f}_{j+1/2}' \right)^T, \quad q_1 = \left( \frac{1}{16} f_0, -\frac{23}{48h} f_0 \right)^T, \quad q_2 = \left( -\frac{1}{54} f_0, 0 \right)^T.
\]

\[
P = \begin{pmatrix}
E & C & \text{ } & \text{ } \\
A & E & B & \text{ } & \text{ } \\
\text{ } & A & E & B & \text{ } \\
\text{ } & \text{ } & A & E & B \\
\text{ } & \text{ } & \text{ } & A & E \\
D & E & \text{ } & \text{ } & \text{ }
\end{pmatrix}, \quad Q = \begin{pmatrix}
b_1 & c_1 & d_1 \\
b & c & \text{ } \\
\text{ } & a & b & c \\
\text{ } & \text{ } & a & b & c \\
\text{ } & \text{ } & \text{ } & a_N & b_N & c_N & d_N
\end{pmatrix}
\]

\[
a = \left( -\frac{1}{54}, 0 \right)^T, \quad b = \left( \frac{133}{108}, -\frac{3}{h} \right)^T, \quad c = \left( \frac{79}{108}, \frac{3}{h} \right)^T,
\]

\[
b_1 = \left( \frac{139}{48}, -\frac{199}{48h} \right)^T, \quad c_1 = \left( -\frac{23}{48}, \frac{107}{48h} \right)^T, \quad d_1 = \left( \frac{1}{48}, -\frac{5}{48h} \right)^T,
\]

\[
a_N = \left( -\frac{1}{36}, -\frac{23}{108h} \right)^T, \quad b_N = \left( \frac{11}{36}, \frac{247}{108h} \right)^T, \quad c_N = \left( -\frac{97}{36}, -\frac{2075}{108h} \right)^T, \quad d_N = \left( \frac{49}{12}, \frac{53}{4h} \right)
\]

The flux function is given by

\[
\tilde{f} = R \bar{F} = R P^{-1} (Qf + q),
\]

where

\[
\tilde{f} = \left( \tilde{f}_{1/2}, \tilde{f}_{3/2}, \ldots, \tilde{f}_{N+1/2} \right)
\]
The conservative approximation for the first derivative is written as

\[ f' = \frac{1}{h} \tilde{Cf} \]  

(9)

where

\[ C = \begin{pmatrix} -1 & 1 \\ \vdots & \vdots \\ -1 & 1 \end{pmatrix} \]

Substituting Eq.(8) and (9) to Eq.(7), the following semidiscrete approximation therefore holds

\[ \frac{\partial u}{\partial t} = - a \frac{1}{h} CRP^{-1}(Qu + q). \]

The stability characteristics of the semidiscrete approximation depend upon the matrix \( S = -CRP^{-1}Q \). In particular, a necessary condition for stability is that all eigenvalues of \( S \) have negative real part and therefore correspond to exponentially damped modes. In order to analyze the stability properties of the scheme (Eq.(4), (5) and (6)) we have numerically computed the eigenvalues of the matrix \( S \) for the number of grid points \( N = 50,100,200,400 \).

The results of the analysis are reported in Fig.1, where we also report the spectrum of \( S \) for the case of periodic boundary conditions. The figure clearly shows that the selected scheme supplemented with appropriate boundary condition is linearly stable regardless of the number of grid points. In addition, the figure shows that the effect of the boundary closures Eq.(5) and (6) is to stabilize the overall algorithm. It is noted that the eigenvalue spectrum in the presence of boundary closures entirely lies inside the region bounded by the eigenvalue spectrum for periodic boundary conditions.
2.3 Coupling the CCD scheme with the WENO scheme

In order to couple CCD with WENO, we adopt the same coupling procedure as in [1], which is based on the evaluation of the absolute value of the difference of the computed solution in the two adjacent nodes, i.e.,

\[ r_j = |v_{j+1} - v_j| \]

When the difference \( r_j \) is larger than a threshold value \( \beta \), the flux function at the intermediate node \( j + 1/2 \) and the two neighboring ones \( (j-1/2, j+3/2) \) are computed by WENO. As \( \beta \) decreases, the effect of WENO on hybrid scheme becomes more dominant. It is noted that when CCD is used, at least two consecutive nodes must be computed by CCD because the CCD algorithm can’t be applied to only one single node. For example, if the nodes at \( j-1/2 \) and \( j+3/2 \) are computed by WENO, \( j+1/2 \) must be also computed by WENO.

The special treatment of CCD is needed at the interface with WENO for smooth coupling. The following scheme is used at the interface:

at the left interface

\[ E \begin{pmatrix} \tilde{f}_{j+1/2} \\ \tilde{f}_{j+3/2} \end{pmatrix} + C_2 \begin{pmatrix} \tilde{f}_{j+3/2} \\ \tilde{f}_{j+3/2} \end{pmatrix} = S_{j+1/2} \]

where
\[ C_2 = \begin{pmatrix} \frac{5}{9} & \frac{1}{12} \\ \frac{8}{9} & -1 \\ \frac{1}{12} & -1 \end{pmatrix}, \quad S_{j+1/2} = \begin{pmatrix} \frac{1}{54} f_{j-1} + \frac{139}{216} f_j + \frac{247}{216} f_{j+1} - \frac{1}{4} f_{j-1/2}^{WENO} \\ \frac{1}{h} \left( -\frac{1}{27} f_{j-1} - \frac{521}{216} f_j + \frac{559}{216} f_{j+1} + \frac{3}{4} f_{j-1/2}^{WENO} \right) \end{pmatrix}, \]

at the right interface
\[ D_2 \begin{pmatrix} f_j^- \\ f_j^+ \end{pmatrix} + E \begin{pmatrix} f_{j+1/2}^- \\ f_{j+1/2}^+ \end{pmatrix} = S_{j+1/2}, \]

where
\[ D_2 = \begin{pmatrix} 1 & h \\ 0 & \frac{1}{4} \end{pmatrix}, \quad S_{j+1/2} = \begin{pmatrix} -\frac{1}{18} f_{j-1} + \frac{131}{72} f_j + \frac{23}{72} f_{j+1} - \frac{1}{12} f_{j_{3/2}}^{WENO} \\ \frac{1}{h} \left( -\frac{1}{9} f_{j-1} - \frac{89}{72} f_j + \frac{127}{72} f_{j+1} + \frac{5}{12} f_{j_{3/2}}^{WENO} \right) \end{pmatrix}, \]

This CCD closure uses the interface fluxes that are pre-determined by WENO.

### 3 NUMERICAL TEST

#### 3.1 Shock-entropy wave interaction in 1D

This test case is proposed by Shu&Osher\(^7\) and has been studied to evaluate performance of schemes widely. The governing equations are the one-dimensional Euler equations. And the initial conditions are
\[ U(x,0) = \begin{cases} (3.857143, 2.629369, 10.33333) & \text{if } 0 \leq x < 1 \\ (1 + 0.2 \sin(5x - 5), 0, 1) & \text{if } 1 \leq x < 60 \end{cases}. \]

Physically, this problem represents that a Mach 3 shock wave interacts with a sine entropy wave that will generate a flow field with both smooth structures and discontinuities. The numerical solution is computed in time up to \( t = 10.8 \). Zero time variation of the conservative variables is specified at the left and right boundaries. Numerical results of the density distribution for \( N = 1200 \) using HCD and HCCD are shown in Fig.2. The hybrid switch parameter is \( \beta = 0.8 \). In Fig.2 we also report the “exact” solution which we have obtained by computing the same problem with \( N = 12000 \). The figure shows that HCCD performs better than HCD. The enlarged portions of Fig.2 around the post-shock regions are presented in Fig.3. It can be seen that HCCD resolve the complex structures in higher resolution.
3.2 Shock-vortex interaction in 2D

This problem is proposed by Chatterjee\(^8\) and has been widely discussed in the literatures. The governing equations are the two-dimensional Euler equations. Initially, a left-moving Mach 1.5 planar shock is placed at \(x = 1.66\). Ambient conditions are prescribed to the left of the shock with the velocity being perturbed by a compressible vortex centered at \((x, y) = (0,0)\). The vortex is modeled
\[ U_\theta (r) = \begin{cases} 
U_c, & r < 0.5 \\
2/3U_c (-r + 1/r), & 0.5 \leq r \leq 1 
\end{cases} \]

where \( U_\theta \) is tangential velocity, \( U_c \) is velocity behind the shock and \( r \) is distance from the vortex center.

The flow variables behind the shock are determined according to the Rankine-Hugoniot relation. The characteristic boundary conditions based on the Riemann invariants are used at left and right boundaries and the periodic conditions are prescribed at the top and bottom boundaries. The computational domain is \([-2,2]\times[-3,3]\), which is discretized into 200\times300 uniform grids. For the purpose of comparing with an “exact” solution, we also compute the same test case with 7th-order WENO scheme on a much finer grid, consisting of 600\times900 nodes. These simulations are performed by HCCD and HCD. The hybrid switch parameter is set to \( \beta = 0.4 \). The density contours at \( t = 1.538 \) are shown in Fig.4, and the enlarged portions of Fig.4 near the vortex center are shown in Fig.5. In these figures, HCCD resolve the density variation at \( x = 0.4 \) better than HCD.
Deng&Zhang\textsuperscript{2} proposed the Weighted Compact Nonlinear Scheme (WCNS) which does not need the switching and can resolve the discontinuity without oscillatory. However, WCNS does not have as high resolution as CD schemes.

We rearrange the WCNS to the finite-volume representation and introduce CCD instead of CD in WCNS, such that a new Weighted Combined Compact Nonlinear Scheme (WCCNS) is obtained. The result of the numerical test shows that WCCNS has better performance than WCNS. The results of the problem in section 3.2 are shown in Fig. 6. The computational domain is $[-2,2] \times [-2,2]$, which is discretized into $500 \times 500$ uniform grid. An “exact” solution is computed with 7th-order WENO scheme on a much finer grid, consisting of $1000 \times 1000$ nodes. Other computational conditions are equal to ones in section 3.2.

The density contours at $t = 2.1$ are showed in Fig. 6. The difference between “exact” and WCCNS is smaller than the difference between “exact” and WCNS. The figure shows that in the case of WCNS, numerically oscillations appear near the top and bottom boundary, but in the case of WCCNS we can not get that oscillation.
5 CONCLUSION

We proposed a conservative hybrid combined compact-WENO scheme with high resolution to obtain the solution of hyperbolic conservation laws. The proposed scheme has better resolution properties than the hybrid compact-WENO scheme. In addition, we show the Weighted Combined Compact Nonlinear Scheme has higher resolution than the Weighted Compact Nonlinear Scheme. These examples show that the use of the conservation form and the combined compact difference schemes is important in the numerical scheme of hyperbolic conservation laws.

REFERENCES